

5.1 – Eigenvalues and Eigenvectors

Consider the equation
$$\begin{bmatrix} 6 & 3 & -8 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{bmatrix} \begin{bmatrix} 8 \\ 0 \\ 1 \end{bmatrix} = 5 \begin{bmatrix} 8 \\ 0 \\ 1 \end{bmatrix}.$$

Definition: If A is an $n \times n$ matrix, then a nonzero vector \mathbf{x} in R^n is called an **eigenvector** of A (or of the matrix operator T_A) if $A\mathbf{x}$ is a scalar multiple of \mathbf{x} ; that is, $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar λ . The scalar λ is called an **eigenvalue** of A (or of T_A), and \mathbf{x} is said to be an **eigenvector corresponding to λ** .

Theorem 5.1.1 If A is an $n \times n$ matrix, then λ is an eigenvalue of A if and only if it satisfies the equation $\det(\lambda I - A) = 0$.

If A is an $n \times n$ matrix, then the **characteristic equation** of A is $\det(\lambda I - A) = 0$.

Definition: The **characteristic polynomial** of an $n \times n$ matrix A results from expanding the characteristic equation of A and has the form

$$p(\lambda) = \lambda^n + c_1\lambda^{n-1} + \dots + c_n.$$

The solution space of the system $(\lambda I - A)\mathbf{x} = \mathbf{0}$ is called the **eigenspace** of A corresponding to λ .

Theorem 5.1.3 If A is an $n \times n$ matrix, the following statements are equivalent.

- a) λ is an eigenvalue of A .
- b) λ is a solution of the characteristic equation $\det(\lambda I - A) = 0$.
- c) The system of equations $(\lambda I - A)\mathbf{x} = \mathbf{0}$ has nontrivial solutions.
- d) There is a nonzero vector \mathbf{x} such that $A\mathbf{x} = \lambda\mathbf{x}$.

#9 find the characteristic equation, the eigenvalues, and bases for the eigenspaces of the matrix.

$$\begin{bmatrix} 6 & 3 & -8 \\ 0 & -2 & 0 \\ 1 & 0 & -3 \end{bmatrix}$$

#19 In each part, find the eigenvalues and the corresponding eigenspaces of the stated matrix operator on R^2 . Use geometric reasoning to find the answers. No computations are needed.

- a. Reflection about the line $y = x$.
 - b. Orthogonal projection onto the x -axis.
 - c. Rotation about the origin through a positive angle of 90° .
 - d. Contraction with factor k ($0 \leq k < 1$)
 - e. Shear in the x -direction by a factor k ($k \neq 0$)
-
-
-
-
-



Theorem 5.1.2 If A is an $n \times n$ triangular matrix (upper triangular, lower triangular, or diagonal), then the eigenvalues of A are the entries on the main diagonal of A .

#14 Find the characteristic equation of the matrix by inspection.

$$\begin{bmatrix} 9 & -8 & 6 & 3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}$$

#15 Find the eigenvalues and a basis for each eigenspace of the linear operator defined by the stated formula.

$$T(x, y) = (x + 4y, 2x + 3y)$$

Theorem 5.1.5 Equivalent Statements (extends Theorem 4.9.8)

If A is an $n \times n$ matrix, then the following statements are equivalent.

- a) A is invertible.
- b) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
- c) The reduced row echelon form of A is I_n .
- d) A is expressible as a product of elementary matrices.
- e) $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix \mathbf{b} .
- f) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix \mathbf{b} .
- g) $\det(A) \neq 0$.
- h) The column vectors of A are distinct and linearly independent.
- i) The row vectors of A are distinct and linearly independent.
- j) The column vectors of A span R^n .
- k) The row vectors of A span R^n .
- l) The column vectors of A form a basis for R^n .
- m) The row vectors of A form a basis for R^n .
- n) A has rank n .
- o) A has nullity 0.
- p) The orthogonal complement of the null space of A is R^n .
- q) The orthogonal complement of the row space of A is $\{\mathbf{0}\}$.
- r) $\lambda = 0$ is not an eigenvalue of A .